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PERTURBATIONS FROM A SOURCE IN A TWO-LAYER ATMOSPHERE BOUNDED BY THE HORIZONTAL TERRESTRIAL SURFACE†

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The problem of internal waves excited by a point source in a two-layer atmosphere is investigated in a linear formulation. The lower layer is bounded by a horizontal surface and, the upper layer is unbounded. It is assumed that the vertical displacements and velocities of the particles vary continuously at the layer boundaries, and that the Brunt-Väisälä frequency is constant in each layer but experiences discontinuities at the common boundary of the layers; the source is situated in the lower layer. The asymptotic behaviour of the perturbations in the lower layer at long times is investigated. The solution is found using integral transforms and is expressed in terms of double integrals of many-valued analytic functions. A transformation is proposed which enables the solution to be expressed as the sum of single integrals. The behaviour of these integrals at long times is found by the stationary-phase method. It is shown that a critical cone exists across which the asymptotic behaviour of the system undergoes a change.

A similar problem has been considered in the past for the case in which the lower layer is unbounded [1]. The case of a three-layer atmosphere with only the middle layer bounded has been studied in [2].

1. STATEMENT OF THE PROBLEM AND THE INTEGRAL REPRESENTATION OF THE SOLUTION

Consider an ideal atmosphere filling three-dimensional space and divided into two layers with constant but different Brunt–Väisälä (BV) frequencies. The unit of time is chosen in such a way that the BV frequency in the upper layer equals unity. The BV frequency in the lower layer is $N < 1$. We take the thickness of the lower layer as the unit of length. The origin of a Cartesian system of coordinates xyz is chosen on the unperturbed interface of the layers, with the z axis pointing in the direction opposite to that of the gravity force; a source is situated at the point $(0, 0, -c)$. The strength $Q(t)$ of the source is a continuously differentiable finite function with support in the interval $[0, T]$. It is assumed that $Q(0) = \dot{Q}(0) = 0$. The vertical deviation of fluid particles from the equilibrium position is expressed in terms of the derivative with respect to z of a function $w(x, y, z, t)$ which is the solution of the problem

$$
\begin{aligned}
&(\Delta w)_{tt} + N^2\Delta_2 w = \frac{\dot{Q}(t)}{4\pi}\delta(x)\delta(y)\delta(z+c), \quad -1<z<0 \\
&(\Delta w)_{tt} + \Delta_2 w = 0, \quad 0<z<+\infty
\end{aligned}
\tag{1.1}
$$

The function w and its first-order partial derivatives are bounded and vary continuously across the layer interface. The initial data are zero. The vertical displacements vanish at the lower boundary, consequently, $w_z(-1) = 0$ (the subscript z denotes differentiation with respect to z).

Taking a Laplace transformation with respect to the variable t, with zero initial conditions, and a Hankel transformation with respect to $r = \sqrt{x^2+y^2}$, we will seek a solution of problem (1.1) in the form

$$
w = \frac{1}{16\pi^3 i}\int_0^{+\infty}\int_{C-i\infty}^{C+i\infty}\varphi(u,p,z)J_0(ru)R(p,t)\,du\,dp \tag{1.2}
$$

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$$R(p,t) = \int_0^T Q(\tau)e^{p(t-\tau)}d\tau \tag{1.3}$$

Put

$$\alpha(p) = \sqrt{1+p^2}/p, \quad \omega(p) = \sqrt{N^2+p^2}/p \tag{1.4}$$

For $z > 0$, the function $\varphi(z)$ satisfies the equation

$$\varphi_{zz} - u^2\alpha(p)^2\varphi = 0, \quad 0 < z < +\infty \tag{1.5}$$

Any solution of Eq. (1.5) that is bounded for $z > 0$, Re $\alpha > 0$, satisfies the equation $\varphi_z(z) + u\alpha\varphi(z) = 0$.

Since the function $\varphi(z)$ is continuous at the point $z = 0$, it follows that in order to determine the function $\varphi(z)$ in the interval $[-1, 0]$, we must solve the following boundary-value problem (construct Green's function)

$$\begin{aligned} \varphi_{zz}(z) - u^2\omega^2\varphi(z) &= (u/p)\delta(z+c) \\ \varphi_z(0) + u\alpha\varphi(0) &= 0, \quad \varphi_z(-1) = 0 \end{aligned} \tag{1.6}$$

The Wronskian of the two solutions of Eq. (1.6)

$$\varphi_1(z) = \mathrm{ch}(u\omega(z+1)), \quad \varphi_2(z) = (\omega\mathrm{ch}(u\omega z) - \alpha\mathrm{sh}(u\omega z))/\omega. \tag{1.7}$$

equals

$$\begin{aligned} -\varphi_2'(-1) &= u\zeta(p) \\ \zeta(p) &= \alpha\mathrm{ch}u\omega + \omega\mathrm{sh}u\omega = (\alpha+\omega)e^{u\omega} + (\alpha-\omega)e^{-u\omega} \end{aligned} \tag{1.8}$$

and the solution of boundary-value problem (1.6) has the form

$$\varphi(u,p,z,c) = \frac{1}{p\zeta(p)}\begin{cases} \varphi_1(z)\varphi_2(-c), & -1 \le z \le -c \\ \varphi_1(-c)\varphi_2(z), & -c \le z \le 0 \end{cases}$$

Substituting this function into Eq. (1.2), we obtain the solution of problem (1.1).

For $z > 0$, this solution may be expressed in the form

$$w = v(r,z,c,t) + v(r,z,2-c,t) \tag{1.9}$$

$$v(r,z,c,t) = \frac{1}{16\pi^3 i}\int_0^{+\infty}\int_{C-i\infty}^{C+i\infty} \psi(u,p,z,c)J_0(ru)R(p,t)\,du\,dp \tag{1.10}$$

$$\psi(u,p,z,c) = (p\zeta(p))^{-1}e^{u(\omega(1-c)-\alpha z)} \tag{1.11}$$

where the function $R(p,t)$ is defined by (1.3)

In what follows we shall confine ourselves to investigating the behaviour of the solution in the upper layer.

2. REPRESENTATION OF THE SOLUTION AS A SUM OF SINGLE INTEGRALS

The poles of the function ψ in formula (1.10) are the zeros of the function $\zeta(p)$ defined by (1.8). Note that, by virtue of Eqs (1.4), $\omega(p)$ and $\zeta(p)$ are even functions and $z(\bar{p}) = \overline{\zeta(p)}$.

If p is a zero of $\zeta(p)$, then $\bar{p}$ and $-p$ are also zeros of this function. There are no zeros anywhere but on the imaginary axis. In fact, let p lie in the fourth quadrant and suppose $\mathrm{Re}\,p > 0$. Then $1/p$ lies in the first quadrant and the points $1/p^2$ and $1 + 1/p^2$ lie in the upper half-plane; hence the point $\alpha(p)$ lies in the first quadrant. Similarly, the point $\omega(p)$ lies the first quadrant. But

$$\left|\frac{\alpha(p)-\omega(p)}{\alpha(p)+\omega(p)}\right| \leq 1, \quad \left|e^{u\omega(p)}\right| > 1$$

and p cannot be a zero of $\zeta(p)$. Then neither can $\bar{p}$ or $-p$ be zeros of $\zeta(p)$, and so, if $\mathrm{Re}\,p \neq 0, p$ cannot be a zero of the function $\zeta(p)$.

Let $p = iq$ and $|q| > 1$. In that case

$$\alpha(iq) = \sqrt{q^2-1}/q, \quad \omega(iq) = \sqrt{q^2-N^2}/q$$
$$\zeta(iq) = 2\alpha(iq)\cos(u\omega(iq)) + 2i\omega(iq)\sin(u\omega(iq))$$

and consequently $\zeta(iq) \neq 0$ for real q with $|q| > 1$.

Consider the inner integral in formula (1.2). Connect the points $[-i, i]$ by a cut along the imaginary axis. The integrand is a regular function in the plane slit in this way. Let C be simple smooth contour around the cut, symmetrical about the real and imaginary axes. Then, by Cauchy's theorem, formula (1.10) may be written in the form

$$v(r, z, c, t) = \frac{1}{16\pi^3 i}\int_0^{+\infty}\int_C \psi(u, p, z, c)J_0(ru)R(p, t)du\,dp \tag{2.1}$$

Let C_i denote the intersection of the contour C with the ith quadrant.

As the point p moves in the positive sense along the are $C_1 + C_2$, the point $-p$ moves in the positive sense along the arc $C_3 + C_4$. The integral along the second arc may be reduced to an integral along the first by substituting $p = -p_1$. At complex-conjugate points the integrand takes complex-conjugate values. As a point describes the contour C_1 in the positive sense, the conjugate point describes C_2 in the negative sense. Taking into consideration that the functions $\alpha(p)$ and $\omega(p)$ are even, we obtain, after interchanging the order of integration in formula (2.1),

$$v(r, z, t, c) = \frac{1}{4\pi^3}\mathrm{Im}\int_0^T Q(\tau)\int_{C_1}\int_0^{+\infty}\psi(u, p, z, c)J_0(ru)\,\mathrm{sh}\,p(t-\tau)\,dp\,du\,d\tau \tag{2.2}$$

Noting that $\mathrm{Re}\,\omega > 0$ on the contour C_1, let us transform the integrand. By (1.10) and (1.11) we have

$$\psi(u, p, z, c) = \frac{1}{p(\alpha+\omega)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\alpha-\omega}{\alpha+\omega}\right)^n e^{-u(\omega(c+2n)+\alpha z)}$$

and hence, using the equality

$$\int_0^{+\infty} e^{-vu}J_0(ru)du = \frac{1}{\sqrt{v^2+r^2}}$$

we can write formula (2.2) in the form

$$v_n = v_{n1} + v_{n2}$$
$$v_{nk} = \mathrm{Im}\int_0^T Q(\tau)\int_{\alpha_k}^{\beta_k}\frac{pf_n(p)\,\mathrm{sh}\,p(t-\tau)}{(1+p^2)\sqrt{R_n(r, z, p)}}dp\,d\tau, \quad k = 1, 2 \tag{2.3}$$

where

$$\alpha_1 = 0, \quad \beta_1 = \alpha_2 = iN, \quad \beta_2 = i$$

$$f_n(p) = (1-\kappa)^n(1+\kappa)^{-(n+1)}, \quad \kappa = \sqrt{(N^2+p^2)/(1+p^2)} \tag{2.4}$$
$$R_n(r, z, p) = (p^2/(1+p^2))r^2 + ((c+2n)\kappa + z)^2$$

Note that

$$R_n(r, z, iN) = z^2 - (Nr/b)^2, \quad b^2 = 1 - N^2$$

We shall call

$$K = \{(r, z): bz = Nr\} \tag{2.5}$$

the critical cone. If $(r, z) \notin K$ and the point p lies in some neighbourhood of the point iN, the integrand in formula (2.3) will be an analytical function of p. The contributions from the endpoint iN to the asymptotic behaviour of the functions v_{n1} and v_{n2} will cancel out when addition is performed. But if $(r, z) \in K$, the point $p = iN$ becomes a critical point. The asymptotic behaviour in that case will be investigated in the next section.

Taking the integral (2.3) with $k = 1$, we change the variable of integration by putting

$$\sqrt{(N^2 + p^2)/(1 + p^2)} = N\sin\varphi, \quad 0 < \varphi < \pi/2$$

and defining

$$B_n = \sqrt{(c + 2n)^2 + \left(\frac{r}{b}\right)^2}, \quad \sin\alpha_n = \frac{r}{bB_n}, \quad \cos\alpha_n = \frac{c + 2n}{B_n} \tag{2.6}$$

and we obtain

$$v_{n1} = \mathrm{Re}\frac{N}{B_n}\int_0^T Q(\tau) \int_0^{\pi/2} A_n^-(\varphi)F_n(\varphi)\sin\varphi\sin(N(t-\tau)A(\varphi))d\varphi d\tau \tag{2.7}$$

where

$$\begin{aligned} F_n(\varphi) &= A_n^+(\varphi)\cos\varphi\frac{(1 - N\sin\varphi)^{n-1}}{(1 + N\sin\varphi)^{n+2}} \\ A(\varphi) &= \frac{\cos\varphi}{\sqrt{1 - N^2\sin^2\varphi}}, \quad A_n^{\pm}(\varphi) = (z/NB_n + \sin(\varphi \pm \alpha_n))^{-1/2} \end{aligned} \tag{2.8}$$

We now transform formula (2.3) for $k = 2$. Making the replacement of variable

$$\sqrt{1 + p^2} = b\cos\varphi, \quad \sqrt{N^2 + p^2} = ib\sin\varphi, \quad 0 < \varphi < \frac{\pi}{2}$$

we obtain

$$v_{n2} = \mathrm{Re}\int_0^T Q(\tau) \int_0^{\pi/2} \frac{D_n(\varphi)\sin\varphi\sin((t-\tau)B(\varphi))}{\sqrt{S_n(r, z, \varphi)}}d\varphi d\tau \tag{2.9}$$

where

$$\begin{aligned} B(\varphi) &= \sqrt{1 - b^2\cos^2\varphi} = \sqrt{N^2 + b^2\sin^2\varphi} \\ S_n(r, z, \varphi) &= (z\cos\varphi + i(c + 2n)\sin\varphi)^2 - (1 - b^2\cos^2\varphi)(r/b)^2 \\ D_n(\varphi) &= e^{-i(2n+1)\varphi}\cos\varphi \end{aligned} \tag{2.10}$$

3. THE ASYMPTOTIC BEHAVIOUR OF THE FUNCTIONS v_{n1} AND v_{n2} AS $t \to +\infty$

We know from the general theory that, up to an arbitrary negative power of t, the main contribution to the asymptotic behaviour comes from the endpoints of the intervals of integration and the stationary points. Let us estimate the contributions of the endpoints. The endpoint $\varphi = 0$ in integrals (2.7) and (2.9) corresponds to the point $p = iN$. As already remarked, these contributions cancel each other out and do not have to be evaluated. It follows from formulae (2.8) that $\varphi = \pi/2$ is not a stationary point of the phase $A(\varphi)$ and that $Fn(\pi/2) = 0$. Consequently, the contribution of the point $\varphi = \pi/2$ to the asymptotic behaviour of the integral v_{n1} decreases as $t \to +\infty$ no more slowly than t^{-2}. It follows from formulae (2.10) that $\varphi = \pi/2$ is a stationary point of the phase and that $D_n(\pi/2) = 0$. The contribution of the point $\varphi = \pi/2$ to the asymptotic behaviour of the integral v_{n1} increases as $t \to +\infty$ no more slowly than $t^{-3/2}$.

Let us estimate the contribution to the expression v_{n1} from the stationary point. It follows from formula (2.7) that the stationary point will lie in the interval $[0, \pi/2]$ only if $z(NB_n) < 1$. Setting

$$\sin\beta_n = z/(NB_n) \tag{3.1}$$

we deduce that the stationary point

$$\gamma_n = \alpha_n - \beta_n \tag{3.2}$$

will belong to the interval $[0, \pi/2]$ only if $bz < Nr$. When this inequality holds, formula (2.7) becomes

$$v_{n1} = \frac{N}{B_n}\int_0^T Q(\tau)\int_{\gamma_n}^{\pi/2}\frac{F_n(\varphi)\sin\varphi\sin(N(t-\tau)A(\varphi))}{\sqrt{\sin(\beta_n + \sin(\varphi-\alpha_n))}}d\varphi d\tau \tag{3.3}$$

The main contribution from the stationary point $\varphi = \gamma_n$ equals the contribution from the point $u = 0$ of the standard integral [3]

$$\begin{aligned}(v_{n1})_0 &\approx \frac{NF_n(\gamma_n)\sin\gamma_n}{B_n\sqrt{\cos\beta_n}}\int_0^T Q(\tau)\int_0^\infty \sin(N(t-\tau)(A(\gamma_n) + A'(\gamma_n)u))\frac{du}{\sqrt{u}} = \\ &= \frac{F_n(\gamma_n)\sqrt{\pi N}\sin\gamma_n}{B_n\sqrt{|A'(\gamma_n)|\cos\beta_n}}\int_0^T \frac{Q(\tau)}{\sqrt{t-\tau}}\sin\left(N(t-\tau)A(\gamma_n) - \frac{\pi}{4}\right)d\tau\end{aligned} \tag{3.4}$$

The functions F_n and A are defined by formula (2.8), $A'(\gamma_n) \neq 0$, the number β_n is defined by formula (3.1), α_n by formula (2.6), and γ_n by formula (3.2).

Note that on the critical cone $bz = Nr$ we have conditions

$$\gamma_n = 0, \quad A'(\gamma_n) = 0$$

The right-hand side of (3.4) vanishes and consequently cannot be the principal term of the asymptotic series. The fact is that on this cone $\varphi = 0$ is a stationary point for the integrals v_{n1} and v_{n2}. Let us evaluate the contributions from the stationary point $\varphi = 0$ in that case. By formulae (2.5), the following conditions hold for $bz = Nr + 0$ and $\varphi \to 0$:

$$\begin{aligned}&\frac{z}{NB_n} + \sin(\varphi - \alpha_n) = \varphi\cos\alpha_n + o(\varphi), \quad A(\varphi) = 1 - \frac{b^2}{2}\varphi^2 + o(\varphi^2) \\ &S_n = 2iz(c+2n)\varphi + o(\varphi) \\ &D_n(\varphi) = 1 + o(1), \quad B(\varphi) = N + \frac{b^2}{2N}\varphi^2 + o(\varphi^2)\end{aligned} \tag{3.5}$$

It follows from formulae (2.7) and (2.9) that in the case the main contribution of the point $\varphi = 0$ equals the standard integrals [2]

$$\begin{aligned}&(v_{nk})_0 = G_{nk}I_{nk}, \quad k = 1, 2 \\ &G_{n1} = \frac{NF_n(0)}{B_n\sqrt{\cos\alpha_n}}, \quad G_{n2} = \mathrm{Re}\frac{1}{\sqrt{2iz(c+2n)}} \\ &I_{nk} = \int_0^T Q(\tau)\int_0^{+\infty}\sqrt{\varphi}\sin\left(N(t-\tau)\left(1 + (-1)^k\frac{b^2}{2}\varphi^2\right)\right)d\varphi d\tau\end{aligned} \tag{3.6}$$

It follows from formulae (2.6), (2.8) and (3.5) that $G_{n1} = G_{n2}$. Evaluating the inner integrals in formulae (3.6), we conclude that the following asymptotic formula holds on the critical cone

$$v_n = (v_{n1})_0 + (v_{n2})_0 = \frac{2(N/2)^{1/2}\cos(3\pi/8)}{B_n b^{3/2}\sqrt{\sin\alpha_n\cos\alpha_n}}\int_0^T \frac{Q(\tau)}{(t-\tau)^{3/4}}\sin N(t-\tau)d\tau$$

Thus, the perturbations in the upper layer from the source in the lower layer alternate as $t \to \infty$. The asymptotic behaviour is not uniform relative to the space variables. Within the critical cone $K = \{(r, z): bz = Nr\}$, the perturbations are of the order $t^{-1/2}$, and outside the critical cone they are of the order $t^{-3/2}$. In the transition zone near the critical cone the perturbations are of the order $t^{-3/4}$.

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